TEMPERATURE FIELD OF A SHORT TUBE WITH INTERNAL HEAT SOURCES

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An approximate solution is obtained for the problem of the temperature field of a short tube with internal heat sources in the steady-state thermal regime.

The problem of the temperature field of an electrical coil of cylindrical shape may be regarded, in idealized form, as the problem of a short tube with internal heat sources.

The temperature field of a tube of finite length with internal heat sources is described by the differential equation

$$\frac{\partial^2 t_{r,y}}{\partial r^2} + \frac{1}{r}\frac{\partial t_{r,y}}{\partial r} + \frac{\partial^2 t_{r,y}}{\partial y^2} + \frac{q_v}{\lambda} = 0. \quad (1)$$

We find the solution of this equation in the form of a sum of two functions [1], each of which depends only on a single variable, i. e., in the form

$$t_{r,y} = F(r) + \Phi(y). \quad (2)$$

Finding the corresponding partial derivatives for (2) and substituting into (1), we obtain

$$F''(r) + \frac{1}{r} F'(r) = -\left[\Phi''(y) + \frac{q_v}{\lambda}\right]. \quad (3)$$

Equation (3) must be satisfied at all values of r and y, which is possible only if both sides are equal to a constant quantity, which we denote by $(-q_V/\lambda\varepsilon)$, where each of the quantities, including ε, is also constant. Then,

$$F''(r) + \frac{1}{r} F'(r) = -\frac{q_v}{\lambda\varepsilon}, \quad (4)$$

$$\Phi''(y) = -\frac{\varepsilon - 1}{\varepsilon}\frac{q_v}{\lambda}. \quad (5)$$

The integrals of these equations have the form

$$\Phi(y) = -\frac{\varepsilon - 1}{\varepsilon}\frac{q_v}{2\lambda} y^2 + C_1 y + C_2 \quad (6)$$

and

$$F(r) = -\frac{q_v}{4\lambda\varepsilon} r^2 + C_3 \ln r + C_4, \quad (7)$$

where C_1, C_2, C_3, C_4, the coefficients of integration, are constant if the conditions of uniqueness attached to this solution are independent of the variables r and y.

In this case, solution (2) is written in the form

$$t_{r,y} = -\frac{q_v r^2}{4\lambda\varepsilon} + C_3 \ln r - \frac{\varepsilon - 1}{\varepsilon}\frac{q_v}{2\lambda} y^2 + C_1 y + C_2 + C_4. \quad (8)$$

The constant ε can be found after determining the coefficients of integration by applying (6) and (7) to the same fixed point of the tube.

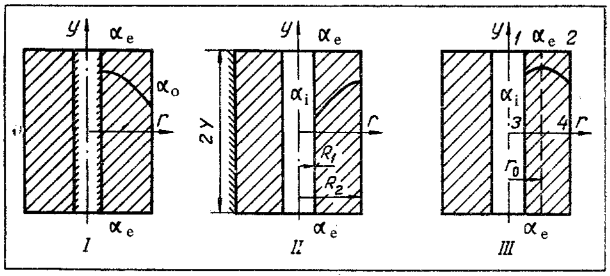

Fig. 1. Assignment of boundary conditions.

The coefficients of integration C_1, C_2, C_3, and C_4 can be found from the conditions of uniqueness, which we formulate for a finite tube in several variants, these conditions being in each case independent of the variables r and y.

I. Cooling under boundary conditions of the third kind from the outer surface and the ends, no heat transfer at the inner surface (Fig. 1) ; in this case,

$$\frac{\partial t_{r,0}}{\partial y} = 0, \tag{9}$$

$$\frac{\partial t_{R_1,y}}{\partial r} = 0, \tag{10}$$

$$-\lambda \frac{\partial t_{R_2,y}}{\partial r} = \alpha_o [t_{(R_2,y)m} - t_a], \tag{11}$$

$$-\lambda \frac{\partial t_{r,Y}}{\partial y} = \alpha_e [t_{(r,Y)m} - t_a]. \tag{12}$$

II. Cooling under boundary conditions of the third kind from the inner surface and the ends, no heat transfer at the outer surface; in this case

$$\frac{\partial t_{r,0}}{\partial y} = 0, \tag{9'}$$

$$\frac{\partial t_{R_2,y}}{\partial r} = 0, \tag{10'}$$

$$-\lambda \frac{\partial t_{R_1,y}}{\partial r} = \alpha_i [t_{(R_1,y)m} - t_a], \tag{11'}$$

$$-\lambda \frac{\partial t_{r,Y}}{\partial y} = \alpha_e [t_{(r,Y)m} - t_a]. \tag{12'}$$

III. Cooling under boundary conditions of the third kind from the outer surface, the inner surface and the ends; in this case,

$$\frac{\partial t_{r,0}}{\partial y} = 0, \tag{9''}$$

$$\frac{\partial t_{r_0,y}}{\partial r} = 0, \tag{10''}$$

$$-\lambda \frac{\partial t_{R_2,y}}{\partial r} = \alpha_o [t_{(R_2,y)m} - t_a], \tag{11''}$$

$$-\lambda \frac{\partial t_{r,Y}}{\partial y} = \alpha_e [t_{(r,Y)m} - t_a], \tag{12''}$$

$$\lambda \frac{\partial t_{R_1,y}}{\partial r} = \alpha_i [t_{(R_1,y)m} - t_a]. \tag{13}$$

In all three cases, the symmetry of the problem with respect to the y-coordinate is expressed by the same derivative (9), (9'), and (9").

In all cases, the boundary conditions are approximate, in that the heat transfer at the surface is expressed in terms of the difference between the mean t of the surface and the surrounding medium.

Let us examine the determination of the coefficients C_1, C_2, C_3, and C_4 for case I.

Using conditions (9) and (10), we determine

$$C_3 = \frac{q_v R_1^2}{2\lambda\varepsilon} \text{ and } C_1 = 0,$$

and solution (8) takes the form

$$t_{r,y} = \frac{q_v}{2\lambda\varepsilon}\left[-\frac{r^2}{2} + R_1^2 \ln r - (\varepsilon - 1)\, y^2\right] + C_2 + C_4. \tag{14}$$

There is also another possibility of determining ε, C_2, and C_4. If we substitute $D = C_2 + C_4$, write (14) in the form

$$t_{r,y} = \frac{q_v}{2\lambda\varepsilon}\left[-\frac{r^2}{2} + R_1^2 \ln r - (\varepsilon - 1)\, y^2\right] + D \tag{15}$$

and do not pose the problem of separate determination of C_2 and C_4, then only the two constants ε and D will be subject to determination; this eliminates the possibility of using (6) and (7) for determining ε, but the boundary conditions (11) and (12) are sufficient for determining both of the constants ε and D.

We determined the values of ε and D from conditions (11) and (12), for which the derivatives and mean temperatures were found using (15). In particular,

$$t_{(R_2,y)\mathrm{m}} = \frac{1}{Y}\int_0^Y t_{R_2,y}\, dy = \frac{q_v}{2\lambda\varepsilon} \times$$

$$\times \left[-\frac{R_2^2}{2} + R_1^2 \ln R_2 - \frac{\varepsilon - 1}{3} Y^2\right] + D,$$

$$t_{(r,Y)\mathrm{m}} = \frac{2}{R_2^2 - R_1^2}\int_{R_1}^{R_2} t_{r,Y}\, r\, dr = \frac{q_v}{2\lambda\varepsilon} \times$$

$$\times \left[-\frac{R_2^2 - 3R_1^2}{4} + \frac{R_1^2 (R_2^2 \ln R_2 - R_1^2 \ln R_1)}{R_2^2 - R_1^2} - (\varepsilon - 1)\, Y^2\right] + D.$$

Having thus solved Eqs. (11) and (12) for ε and D, we obtain

$$\varepsilon = 1 + \frac{3\lambda\alpha_e}{\alpha_e Y^2 + 3\lambda Y}\left\{\frac{R_2}{2\alpha_0}\left[1 - \left(\frac{R_1}{R_2}\right)^2\right] + \frac{R_2^2}{\lambda} \times \right.$$

$$\left. \times \left[\frac{1}{8} - \frac{3}{8}\left(\frac{R_1}{R_2}\right)^2 - \left(\frac{R_1}{R_2}\right)^4 \frac{\ln \dfrac{R_1}{R_2}}{2\left[1 - \left(\dfrac{R_1}{R_2}\right)^2\right]}\right]\right\}, \tag{16}$$

$$D = \frac{q_v}{2\lambda\varepsilon}\left[\frac{\lambda (R_2^2 - R_1^2)}{\alpha_0 R_2} + \frac{R_2^2 - 2R_1^2 \ln R_2}{2} + \frac{\varepsilon - 1}{3} Y^2\right] + t_a. \tag{17}$$

Substituting (17) into (15), we have

$$t_{r,y} = \frac{q_v}{2\lambda\varepsilon}\left[-\frac{r^2}{2} + R_1^2 \ln r - (\varepsilon - 1)\, y^2 + \frac{\lambda (R_2^2 - R_1^2)}{\alpha_0 R_2} + \right.$$

$$\left. + \frac{R_2^2 - 2R_1^2 \ln R_2}{2} + \frac{\varepsilon - 1}{3} Y^2\right] + t_a. \tag{18}$$

Then, Eq, (18), for which the value of ε is determined from (16), is the solution of the problem.

We have borrowed certain details of our approach to the solution of the problem from [3], which is concerned with the question of the temperature field of a cylinder of finite length with internal heat sources.

Similarly, for conditions (9'), (10'), (11'), and (12') we obtained a solution in the form

$$t_{r,y}=\frac{q_v}{2\lambda\varepsilon}\left[-\frac{r^2}{2}+R_2^2\ln r-(\varepsilon-1)y^2-\frac{\lambda(R_2^2-R_1^2)}{\alpha_i R_1}+\right.$$
$$\left.+\frac{R_1^2-2R_2^2\ln R_1}{2}+\frac{\varepsilon-1}{3}Y^2\right]+t_a, \tag{19}$$

$$\varepsilon=1+\frac{3\lambda\alpha_e}{\alpha_e Y^2+3\lambda Y}\left\{\frac{R_1}{2\alpha_i}\left[1-\left(\frac{R_2}{R_1}\right)^2\right]+\frac{R_1^2}{\lambda}\times\right.$$
$$\left.\times\left[\frac{1}{8}-\frac{3}{8}\left(\frac{R_2}{R_1}\right)^2+\left(\frac{R_2}{R_1}\right)^4\frac{\ln\frac{R_2}{R_1}}{2\left[\left(\frac{R_2}{R_1}\right)^2-1\right]}\right]\right\}. \tag{20}$$

Finally, for conditions (9"), (10"), (11"), (12"), and (13) the solution takes the form

$$t_{r,y}=\frac{q_v}{2\lambda\varepsilon}\left[-\frac{r^2}{2}+r_0^2\ln r-(\varepsilon-1)y^2+\frac{\lambda(R_2^2-r_0^2)}{\alpha_o R_2}+\right.$$
$$\left.+\frac{R_2^2-2r_0^2\ln R_2}{2}+\frac{\varepsilon-1}{3}Y^2\right]+t_a, \tag{21}$$

$$\varepsilon=1+\frac{3\lambda\alpha_e}{\alpha_e Y^2+3\lambda Y}\left\{\frac{R_2}{2\alpha_o}\left[1-\left(\frac{r_0}{R_2}\right)^2\right]+\frac{R_2^2}{\lambda}\times\right.$$
$$\left.\times\left[\frac{1}{8}-\frac{1}{8}\left(\frac{R_1}{R_2}\right)^2-\frac{1}{4}\left(\frac{r_0}{R_2}\right)^2-\left(\frac{R_1}{R_2}\right)^2\left(\frac{r_0}{R_2}\right)^2\frac{\ln\frac{R_1}{R_2}}{2\left[1-\left(\frac{R_1}{R_2}\right)^2\right]}\right]\right\}, \tag{22}$$

$$r_0^2=\frac{R_1R_2[2\lambda(\alpha_o R_1+\alpha_i R_2)+\alpha_i\alpha_o(R_2^2-R_1^2)]}{2\lambda(\alpha_o R_2+\alpha_i R_1)+2\alpha_i\alpha_o R_1R_2\ln\frac{R_2}{R_1}}. \tag{23}$$

The solutions obtained satisfy the differential equation (1) and the corresponding equations of uniqueness (9), (10), (11), (12); (9'), (10'), (11'), (12'); (9"), (10"), (11"), (12"), (13).

At $Y=\infty$, the solutions go over into the equations of the temperature field of an infinite tube with internal heat sources [2].

At $R_2=\infty$ the solution go over into the equations of the temperature field of an infinite plate with internal heat sources [2].

Thus, at $Y=\infty$ or $R_2=\infty$ the solutions are exact. This is perfectly legitimate, since the mean surface temperatures entering into the boundary conditions then become the true temperatures, and the boundary conditions themselves become not approximate, but exact.

At finite values of Y and R_2 the solutions are approximate owing to the simplification of the boundary conditions and the consequent approximation of the constants ε and D and C_1 and C_3.

At $\alpha_i=0$ Eqs. (19) and (20) go over into the solution for an infinite plate, at $\alpha_e=0$ into the solution for an infinite tube cooled only over the inner surface.

Equations (21), (22), and (23) are the most general solution and at $\alpha_i=0$ go over to Eq. (18) and (16); at $\alpha_0=0$ they are transformed into Eqs. (19) and (20); at $\alpha_i=0$ and $\alpha_0=0$ they become the solution for an infinite plate; at $\alpha_i=0$ and $\alpha_e=0$ they are transformed into the solution for a finite tube cooled only over the outer surface; at $\alpha_e=0$

and $\alpha_0 = 0$ they go over to the solution for an infinite tube cooled only over the inner surface; and at $\alpha_e = 0$ they become the solution for an infinite tube cooled at the outer and inner surfaces.

The suitability of the method was tested on several examples for the most general case—a short tube with internal heat sources and simultaneous cooling of the inner, outer, and end surfaces. The starting data are given in Table 1, and the results are compared in Fig. 2.

Table 1. Starting Data for the Examples ($q_V = 10^5$ W/m^3, $\lambda = 0.4$ W/m · deg, $a = 2 \cdot 10^{-7}$ m^2/sec)

	Tube dimensions in mm: a	b	c	Heat transfer coefficients, W/m² · deg
R_1	5	5	5	$\alpha_i=5$
R_2	17	15	9	$\alpha_o=20$
$2Y$	32	4	32	$\alpha_e=16$

($q_v=10^5$ W/m³, $\lambda=0.4$ W/m · deg, $a=2 \cdot 10^{-7}$ m²/sec)

The temperature fields for all the examples were calculated from (23), (22), and (21) on a "Supermetall" electric calculating machine, and by the method of elementary balances using a Minsk-1 computer.

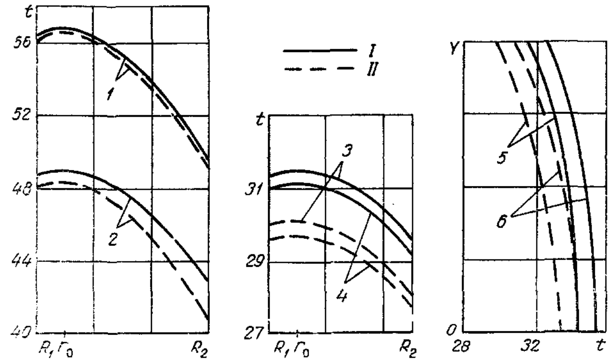

Fig. 2. Comparison of calculations based on Eqs. (23), (22), and (21) and computer calculations based on the method of elementary balances: I) computer calculation; II) calculations based on the formulas; 1 and 2) example a $t_{y=0}$ and $t_{y=Y}$; 3 and 4) example b $t_{y=0}$ and $t_{y=Y}$; 5 and 6) example c t_{R_2} and t_{R_1}; t in °C.

The similar character of the temperature curves obtained from equations (23), (22), (21) and by the method of elementary balances is apparent from Fig. 2. The radii r_0, at which the temperature maximum in the radial direction is established, coincide. The greatest errors are observed at the junctions of the ends and the cylindrical surfaces, which is quite natural in view of the simplified boundary conditions. Table 2 presents the errors δ (%) of the calculations based on (23), (22), and (21) for the characteristic points 1, 2, 3, and 4 (Fig. 1, III) relative to the method of elementary balances, whose errors for examples a and b were taken into account using the Runge formula [4] (the calculations were made for networks with steps 2h and h). In example c, the error was computed without allowance for the advantages that can be obtained in these calculations by refining the network.

Table 2. Calculation Errors

Example	a				b				c			
No. of points	1	2	3	4	1	2	3	4	1	2	3	4
δ, %	−1.05	−4.7	−0.25	−0.12	−4.3	−5.1	−4.2	−4.8	−5.4	−6.0	−3.4	3.0

As may be seen from the table, the accuracy of the solution is satisfactory. It should be noted that, in these problems, when the boundary conditions of the third kind are strictly formulated, the coefficients of integration obviously depend only slightly on the variables r and y and, therefore, are not seriously affected by adopting approximate boundary conditions.

The relative simplicity, generality and satisfactory accuracy of the solution make it possible to recommend it for calculating the temperature regimes of electrical coils. It is desirable to take into account the effect of the coil frame as a factor affecting α_i, α_0, and α_e; however, the question of the distribution of the total power of the coil between the inner and outer cylindrical and end surfaces (i. e., the question of the magnitude of α_i, α_0, and α_e) is not considered in this study.

NOTATION

$t_{r,y}$ is the temperature at point with coordinates r and y; r and y are variable coordinates in the cylindrical system; R_1, R_2, and 2Y are the inner and outer radii and the length of the tube, respectively; q_V is the power of the internal heat sources, uniform over the volume; λ is the thermal conductivity of the tube material (does not depend on temperature); α_0, α_i and α_e are the coefficients of heat transfer from the outer and inner cylindrical surfaces and the ends of tube; t_a is the temperature of surrounding medium.

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